Math 6261 2023-02-10

Review:

Def. ① A collection $\mathcal{A}$ of subsets of Ω is said to be a π-system if

$$A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}.$$

② A collection $\mathcal{L}$ of subsets of Ω is said to be a λ-system if

(i) $\Omega \in \mathcal{L}$; (ii) If $A, B \in \mathcal{L}$, $A \subset B$, then $B \setminus A \in \mathcal{L}$;
(iii) If $A_n \in \mathcal{L}$ and $A_n \uparrow A$, then $A \in \mathcal{L}$.

Thm (Dynkin's π-λ Thm)

Suppose that $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system such that

$$\mathcal{P} \subset \mathcal{L}.$$

Then $\sigma(\mathcal{P}) \subset \mathcal{L}$.

Thm 2.2. In order for $X_1, \cdots, X_n$ to be independent, it is sufficient that for all $a_1, \cdots, a_n \in (-\infty, \cdots, +\infty]$,

$$P(X_1 \le a_1, \cdots, X_n \le a_n) = \prod_{i=1}^{n} P(X_i \le a_i). \quad (*)$$

Pf. Let $\mathcal{A}_i$ be the collection of sets of the form $X_i^{-1}(-\infty, a_i]$, $i = 1, \cdots, n$

Then $\mathcal{A}_i$ is a π-system for each i.

By $(*)$, $\mathcal{A}_1, \cdots, \mathcal{A}_n$ are independent. So by Thm 2.1,

$\sigma(\mathcal{A}_1), \cdots, \sigma(\mathcal{A}_n)$ are independent.

But $\sigma(\mathcal{A}_i) = \sigma(X_i)$, so

$X_1, \cdots, X_n$ are independent. $\boxed{/\!/}$

Thm 2.3. Suppose $\mathcal{F}_{i,j}$, $1 \le i \le n$, $1 \le j \le m(i)$ are independent and let $\mathcal{G}_i = \sigma\left(\bigcup_{j=1}^{m(i)} \mathcal{F}_{i,j}\right)$. Then $\mathcal{G}_1, \cdots, \mathcal{G}_n$ are independent.

Pf. Let $\mathcal{A}_i$ be the collection of sets of the form

$$\bigcap_{j=1}^{m(i)} A_{i,j} \quad \text{with } A_{i,j} \in \mathcal{F}_{i,j}.$$

Then $\mathcal{A}_i$ is a π-system for each i.

Notice that $\mathcal{A}_1, \cdots, \mathcal{A}_n$ are independent, so

$\mathcal{G}_1, \cdots, \mathcal{G}_n$ are independent.

Thm 2.4. If $X_{i,j}$, $i = 1, \cdots, n$, $1 \le j \le m(i)$ are independent and

$f_i : \mathbb{R}^{m(i)} \to \mathbb{R}$ are measurable, then

$f_i(X_{i,1}, \cdots, X_{i,m(i)})$ are independent.

Pf. Let $\mathcal{F}_{i,j} = \sigma(X_{i,j})$. and $\mathcal{G}_i = \sigma(\bigcup_{j=1}^{m(i)} \mathcal{F}_{i,j})$.

Then $f_i(X_{i,1}, \cdots, X_{i,m(i)}) \in \mathcal{G}_i$ (**) .

To see (**), it suffices to show that for each $A \in \mathcal{B}(\mathbb{R}^{m(i)})$,

$$(X_{i,1}, \cdots, X_{i,m(i)})^{-1} A \in \mathcal{G}_i. \quad (***)$$

Write

$$\mathcal{T} := \{ A \in \mathcal{B}(\mathbb{R}^{m(i)}) : (X_{i,1}, \cdots, X_{i,m(i)})^{-1} A \in \mathcal{G}_i \}.$$

Clearly, $\mathcal{T}$ is a sub σ-algebra of $\mathcal{B}(\mathbb{R}^{m(i)})$ and $\mathcal{T}$ contains the connection of sets of the form

$$A_1 \times \cdots \times A_{m(i)} \quad \text{with} \quad A_j \in \mathcal{B}(\mathbb{R}).$$

Since this collection generates $\mathcal{B}(\mathbb{R}^{m(i)})$, it follows that

$$\mathcal{T} = \mathcal{B}(\mathbb{R}^{m(i)}),$$

which implies (***). $\blacksquare$

Next we would like to investigate the distribution and expectation of independent r.v.'s.

Thm 2.5. Suppose $X_1, \cdots, X_n$ are independent r.v.'s and X_i has distribution μ_i, then $(X_1, \cdots, X_n)$ has distribution $\mu_1 \times \cdots \times \mu_n$.

Pf. Write
$$\mathcal{T} = \{ A \in \mathcal{B}(\mathbb{R}^n) : P((X_1, \cdots, X_n) \in A) = \mu_1 \times \cdots \times \mu_n (A) \}.$$

Clearly $\mathcal{T}$ is a λ-sytem. Notice that

$$\mathcal{T} \supseteq \underbrace{\{ A_1 \times \cdots \times A_n : A_1, \cdots, A_n \in \mathcal{B}(\mathbb{R}) \}}_{\text{a } \pi\text{-system}}$$

Hence
$$\mathcal{T} \supset \sigma(\{ A_1 \times \cdots \times A_n : A_1, \cdots, A_n \in \mathcal{B}(\mathbb{R}) \})$$
$$= \mathcal{B}(\mathbb{R}^n). \quad \boxtimes.$$

Thm 2.6. Suppose X, Y are independent with distributions μ and ν. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be measurable so that either $h \geq 0$ or
$$\int |h| \, d\mu \times \nu < \infty, \text{ then}$$
$$E\, h(X,Y) = \int h \, d\mu \times \nu = \iint h(x,y) \, d\mu(x) \, d\nu(y).$$

Pf. By Thm 2.5, (X, Y) has distribution $\mu \times \nu$. Then by the change variable formula,
$$E\, h(X,Y) = \int h \, d(\mu \times \nu)$$
$$\overset{\text{Fubini}}{=} \iint h(x,y) \, d\mu(x) \, d\nu(y) \quad \boxtimes$$

Cor. 2.7. Suppose $X_1, \dots, X_n$ are independent such that either

(a) $X_i \geq 0$ for $0 \leq i \leq n$ or

(b) $E|X_i| < \infty$.

Then $E(X_1 \cdots X_n) = \prod_{i=1}^{n} E X_i$. $\square$

Next we consider sums of independent r.v.'s.

Thm 2.8: Let X, Y be independent, $F(x) = P(X \leq x)$, $G(y) = P(Y \leq y)$.

Then for each $z \in \mathbb{R}$,

$$P(X+Y \leq z) = \int F(z-y)\, dG(y).$$

"The convolution of F and G"

Pf. Set $h(x,y) = \mathbb{1}_{\{x+y \leq z\}}$. Then

$$\begin{aligned} P(X+Y \leq z) &= E\, h(X,Y) \\ &= \iint \mathbb{1}_{\{x+y \leq z\}}\, d\mu(x)\, d\nu(y) \\ &= \iint \mathbb{1}_{\{x \leq z-y\}}\, d\mu(x)\, d\nu(y) \\ &= \int \mu(-\infty, z-y]\, d\nu(y) \\ &= \int F(z-y)\, d\nu(y). \end{aligned}$$

$\square$

Remark: $\int f(y)\, dG(y)$ means $\int f(y)\, d\nu(y)$.

<u>Application</u>: Let X, Y be independent r.v.'s

Suppose X has density f and Y has density g.

Then

$X+Y$ has density

$$f * g(z) = \int f(z-y)\, g(y)\, dy.$$

Construction of independent r.v's:

Here we construct a sequence of independent r.v.'s $X_1, X_2, \cdots$

Let $\nu_1, \cdots, \nu_n, \cdots$, be a sequence of Borel probability measures on $\mathbb{R}$.

Define

$$\mathbb{R}^{\mathbb{N}} = \{ (\omega_1, \cdots, \omega_n, \cdots) : \omega_i \in \mathbb{R} \},$$ where $\mathbb{N} := \{1, 2, \cdots\}$

For $\omega = (\omega_i)_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$, define

$$X_i(\omega) = \omega_i.$$

Let $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$ be the σ-algebra generated by the finite dimensional sets $\{\omega : \omega_i \in B_i, 1 \le i \le n\}$ with $B_i \in \mathcal{B}(\mathbb{R})$.

Thm (Kolmogrov's extension thm) Let μ_n be prob. measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, $n = 1, 2, \cdots$, such that

$$\mu_{n+1}\Big((a_1,b_1]\times\cdots\times(a_n,b_n]\times\mathbb{R}\Big)=\mu_n\Big((a_1,b_1]\times\cdots\times(a_n,b_n]\Big)$$

Then $\exists$ a unique prob. measure P on $(\mathbb{R}^{\mathbb{N}},\mathcal{B}(\mathbb{R}^{\mathbb{N}}))$ such that

$$P\Big((a_1,b_1]\times\cdots\times(a_n,b_n]\times\mathbb{R}^{\mathbb{N}}\Big)=\mu_n\Big((a_1,b_1]\times\cdots\times(a_n,b_n]\Big).$$

Set $\mu_n=\nu_1\times\cdots\times\nu_n$. Write

$$P=\prod_{n=1}^{\infty}\nu_n$$

Then $X_1(\omega),\cdots,X_n(\omega),\cdots$ are independent.

§ 2.2 Weak law of large numbers.

Def. We say X_n converges to X if for each $\varepsilon>0$

$$\lim_{n\to\infty}P(|X_n-X|>\varepsilon)=0.$$

Def. A family of r.v.'s, X_i, $i\in\mathcal{I}$, are said to be uncorrelated if

$$E(X_iX_j)=EX_i\,EX_j \quad \text{for} \quad i\neq j.$$

<u>Lemma</u>: • Suppose $X_1,\cdots,X_n$ are uncorrelated with $E(X_i^2)<\infty$.

Then $\operatorname{Var}(X_1+\cdots+X_n)=\operatorname{Var}(X_1)+\cdots+\operatorname{Var}(X_n)$,

where $\operatorname{Var}(X)=E((X-\mu)^2)=E(X^2)-\mu^2$.

- $\mathrm{Var}(cX) = c^2\,\mathrm{Var}(X)$.

<u>Lemma</u>. Let $p>0$. Suppose $E|Z_n|^p \to 0$ then $Z_n \to 0$ in prob.

Pf. Let $\varepsilon>0$. Then by the Chebyshev inequality

$$P\{\,|Z_n| \geq \varepsilon\,\} \leq \frac{E|Z_n|^p}{\varepsilon^p} \to 0 \quad \text{as } n\to\infty.$$

$\boxed{/\!/}$

Thm 2.9 (L^2 weak law of large numbers).

Let $X_1, X_2, \cdots,$ be uncorrelated r.v.'s with $EX_i = \mu$ and $\mathrm{Var}(X_i) \leq C < \infty$.

Then $\dfrac{X_1+\cdots+X_n}{n} \to \mu$ in prob.

Pf. Set $S_n = X_1+\cdots+X_n$.

Let $\varepsilon>0$. Then

$$P\left(\left|\frac{S_n}{n}-\mu\right| \geq \varepsilon\right) \leq \frac{E\left(\frac{S_n}{n}-\mu\right)^2}{\varepsilon^2}$$

$$= \frac{1}{n^2\varepsilon^2}\cdot \mathrm{Var}(X_1+\cdots+X_n)$$

$$= \frac{1}{n^2\varepsilon^2}\left(\mathrm{Var}(X_1)+\cdots+\mathrm{Var}(X_n)\right)$$

$$\leq \frac{C}{n\varepsilon^2} \to 0 \quad \text{as } n\to\infty.$$

$\boxed{/\!/}$

§ 2.3 Borel-Cantelli lemmas

Let $A_n \subset \Omega$, $n=1,2,\cdots$

Write

$$\limsup A_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n$$

$$= \{ \omega : \omega \in A_n \text{ i.o.} \}$$

Thm (Borel-Cantelli Lemma) Suppose $\sum_{n=1}^{\infty} P(A_n) < \infty$.

Then $P(A_n \text{ i.o.}) = 0$

Pf. Since $\sum_{n=1}^{\infty} P(A_n) < \infty$,

$$\lim_{m\to\infty} \sum_{n=m}^{\infty} P(A_n) = 0$$

So $P\left(\bigcup_{n=m}^{\infty} A_n\right) \leq \sum_{n=m}^{\infty} P(A_n) \to 0$ as $m \to \infty$

Hence $P(\limsup A_n) \leq P\left(\bigcup_{n=m}^{\infty} A_n\right) \to 0$ as $m \to \infty$.

$\blacksquare$

As another application of the Borel-Cantelli Lemma, we have the following version of strong law of large numbers.

Thm 2.11. Let $X_1, X_2, \cdots,$ be i.i.d. with $EX_i = \mu$ and $EX_i^4 < \infty$.

Then $$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \mu \quad \text{a.s.}$$